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Abstract. The concept of a universal T matrix, recently introduced by Fronsdal and Galindo in the framework of quantum groups, is discussed here as a generalization of the exponential mapping. New examples related to inhomogeneous quantum groups of physical interest are developed, the duality calculations are explicitly presented and it is found that in some cases the universal T matrix, as for Lie groups, is expressed in terms of usual exponential series.

1. Introduction

The relations of the quantum algebra $Fun_q (SU(2))$ generated by the elements a, b, c and d have the remarkable property of being preserved under matrix multiplication [1]. This means that if we define

$$T' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \qquad T'' = \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}$$

where (a', b', c', d') and (a'', b'', c'', d'') are two mutually commuting sets of elements satisfying the relations of $Fun_q (SU(2))$, then the variables (a, b, c, d) defined by

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv T' T''$$

also satisfy the same relations. This property can be formalized by defining a co-multiplication Δ according to

$$\Delta(T) = T \dot\otimes T \tag{1}$$

where $\dot\otimes$ denotes matrix multiplication and tensor product of the C^*-algebras of the non-commutative representative functions. [2]. For $q = 1$ the matrix $\Delta(T)$, with elements in $\otimes^2 Fun (SU(2))$, gives rise to the ordinary group composition. The inverse matrix then defines a second operation on the elements (a, b, c, d), namely the antipode

$$S(T) = T^{-1} . \tag{2}$$

The antipode and co-multiplication together with co-unit, unit and multiplication are collected into the Hopf algebra $Fun_q (SU(2))$.

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These properties can be abstracted and generalized without referring to matrices. To show this, we consider the quantization $U_q(g)$ of the universal enveloping algebra of a Lie algebra g with product, coproduct and antipode defined by

$$m(X_A \otimes X_B) = f_{AB}^C X_C \qquad \Delta(X_A) = h_A^{BC} X_B \otimes X_C \qquad S(X_A) = s_A^B X_B$$

where the sum over repeated indices is assumed and where X_A are the elements of a basis of $U_q(g)$, as, for instance, the Poincaré–Birkhoff–Witt basis. If g is a compact form of a semisimple Lie algebra, it is well known that the Tannaka theory establishes a duality between the universal enveloping algebra and the Hopf algebra of the representative functions. This kind of duality has also been studied at the quantum level [2] so that we can speak of a compact quantum group $Fun_q(G)$ satisfying the relations

$$m^*(x^C) = f_{AB}^C x^A \otimes x^B \qquad \Delta^*(x^B \otimes x^C) = h_A^{BC} x^A \qquad S^*(x^B) = (s^{-1})_A^B x^A$$

where $\{x^A\}$ is a basis of $Fun_q(G)$ such that $\langle x^A, X_B \rangle = \delta_B^A$. For non-compact Lie algebras the duality is more delicate and the functions vanishing at infinity must be determined [3].

In this scheme, independent of the representation, the object that takes the place of the matrix T, called the universal T matrix and denoted by the same letter, is given by summing the tensor products of all the corresponding elements of a pair of dual bases [4]:

$$T = x^A \otimes X_A .$$

The structure of T and the Hopf algebra operations naturally suggest two kinds of mappings, the first one using the multiplication of $U_q(g)$,

$$T \dot{\otimes} T = (x^A \otimes x^B) \otimes m(X_A \otimes X_B) \tag{3}$$

the other being obtained from the multiplication of $Fun_q(G)$,

$$T \ddot{\otimes} T = \Delta^*(x^A \otimes x^B) \otimes (X_A \otimes X_B) . \tag{4}$$

It is straightforward to see that the duality relations yield the equalities

$$m^*(x^A) \otimes X_A = T \dot{\otimes} T \qquad x^A \otimes \Delta(X_A) = T \ddot{\otimes} T \tag{5}$$

and

$$S^*(x^A) \otimes X_A = x^A \otimes S(X_A) = T^{-1} \tag{6}$$

where T^{-1} is defined so that

$$\Delta^*(T \dot{\otimes} T^{-1}) = \Delta^*(T^{-1} \dot{\otimes} T) = 1 \qquad \text{and} \qquad m(T \ddot{\otimes} T^{-1}) = m(T^{-1} \ddot{\otimes} T) = 1 .$$

For the sake of clarity let us consider the explicit example of a compact Lie algebra g with corresponding Lie group G and representative functions $Fun(G)$. If X_k, $(k = 1, \ldots n)$, are the generators of the Lie algebra, a basis of the universal enveloping algebra is of the form $X_A = X_1^{a_1} X_2^{a_2} \ldots X_n^{a_n}$. The dual elements $x^A \in Fun(G)$ are then $x^A =$

$x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n} / (a_1! a_2! \ldots a_n!)$. where $\langle x_k, X_\ell \rangle = \delta_{k\ell}$. Therefore the universal T matrix results in

$$T = \sum_{a_1} \frac{x_1^{a_1} \otimes X_1^{a_1}}{a_1!} \cdots \sum_{a_n} \frac{x_n^{a_n} \otimes X_n^{a_n}}{a_n!} = e^{x_1 \otimes X_1} \ldots e^{x_n \otimes X_n} .$$

It appears that the evaluation of T on an element of the group G reproduces that element expressed by means of the exponential mapping between g and G, and therefore in the case of Lie groups the universal T matrix can be regarded as a resolution of the identity mapping of G into itself. This point of view must be slightly modified in quantization: recalling that the evaluation determines a character on the algebra of the representative functions, we see that this character reproduces itself when applied to the universal T matrix, despite the fact that x^A are now elements of a non-commutative C^*-algebra.

In the standard framework [1] the quantum relations are obtained using the matrices $T \otimes 1$ and $1 \otimes T$. Accordingly we define

$$T_1 = x^A \otimes (X_A \otimes 1) \qquad T_2 = x^B \otimes (1 \otimes X_B)$$

so that the products $T_1 T_2$ and $T_2 T_1$ read

$$T_1 T_2 = \Delta^*(x^A \otimes x^B) \otimes (X_A \otimes X_B) \qquad T_2 T_1 = \Delta^*(x^B \otimes x^A) \otimes (X_A \otimes X_B)$$

and, as shown before, they can be expressed in the form

$$T_1 T_2 = x^C \otimes \Delta(X_C) \qquad T_2 T_1 = x^C \otimes \sigma \Delta(X_C)$$

where $\sigma(X \otimes Y) = Y \otimes X$. We then see that when an R-matrix does exist, its defining property $R \Delta R^{-1} = \sigma \Delta$ gives immediately the algebraic relation

$$R \, T_1 \, T_2 = T_2 \, T_1 \, R \tag{7}$$

which, when represented, reproduces the well known quantization prescription, [2].

In the next section we briefly summarize the results for the quantum group $SU_q(2)$, which, up to minor additions on the antipode, are contained in [1]. The purpose for doing so is two-fold: first we find it useful to give a developed example of the way in which the universal T matrix works; secondly we want to establish explicit relations that will be relevant to discuss the universal T matrix for some inhomogeneous quantum groups that are related to $SU_q(2)$ and useful for physical applications, namely $H_q(1)$ [5], $E_q(2)$ [6] and $\Gamma_q(1)$ [7]. These will be presented in subsequent sections where we shall see that the T operator, expressed in terms of q-exponentials for $SU_q(2)$, in some cases and in an appropriate basis is simply given by a product of exponentials.

2. The universal T matrix for $SU_q(2)$

Starting from the usual generators J_+, J_- and J_3 of $SU_q(2)$ we define

$$E = e^{z \, J_3/2} \, J_+ \qquad F = e^{-z \, J_3/2} \, J_-$$

that satisfy the commutation relations

$$[J_3, E] = E \qquad [J_3, F] = -F \qquad [E, F] = \frac{2 \sinh (z J_3)}{1 - e^{-z}}.$$

The coproduct and antipodes now have the form

$$\Delta E = 1 \otimes E + E \otimes e^{z J_3} \qquad \Delta F = e^{-z J_3} \otimes F + F \otimes 1 \qquad \Delta J_3 = 1 \otimes J_3 + J_3 \otimes 1$$

$$\tag{8}$$

$$S(E) = -e^{-z J_3} E \qquad S(F) = -e^{z J_3} F \qquad S(J_3) = -J_3.$$

In order to find the T operator we must determine the dual $Fun_q (SU(2))$. We thus begin by defining the elements ϕ, γ, η dual to the generators E, F, J_3 satisfying

$$\langle \phi, F \rangle = 1 \qquad \langle \gamma, J_3 \rangle = 1 \qquad \langle \eta, E \rangle = 1. \tag{9}$$

The three elements ϕ, γ, η generate $Fun_q (SU(2))$ as an algebra [1] and satisfy

$$[\phi, \eta] = 0 \qquad [\gamma, \phi] = -z \phi \qquad [\gamma, \eta] = -z \eta. \tag{10}$$

The coproducts and antipodes are

$$m^*(\phi) = \phi \otimes 1 + (e^{-\gamma/2} \otimes \phi)(1 \otimes 1 + \eta \otimes \phi)^{-1}(e^{-\gamma/2} \otimes 1)$$

$$m^*(\eta) = 1 \otimes \eta + (1 \otimes e^{-\gamma/2})(1 \otimes 1 + \eta \otimes \phi)^{-1}(\eta \otimes e^{-\gamma/2})$$

$$\tag{11}$$

$$m^*(\gamma) = 1 \otimes \gamma + \gamma \otimes 1 - 2z \sum_n \frac{(-\eta \otimes \phi)^n}{1 - e^{-z n}}.$$

$$S^*(\phi) = -(e^{-\gamma/2} + \eta \, e^{\gamma/2}\phi)^{-1} \, e^{\gamma/2}\phi \qquad S^*(\eta) = -\eta \, e^{\gamma/2}(e^{-\gamma/2} + \eta \, e^{\gamma/2}\phi)^{-1}$$

$$\tag{12}$$

$$S^*(e^{\gamma/2}) = e^{-\gamma/2} + \eta \, e^{\gamma/2}\phi.$$

The elements

$$X_{jk\ell} = F^j J_3^k E^\ell$$

define a basis of $U_q(su(2))$. A direct verification shows that the corresponding dual basis in $Fun_q (SU(2))$ is given by

$$x^{jk\ell} = \frac{\phi^j}{[j]_z!} \frac{\gamma^k}{k!} \frac{\eta^\ell}{[\ell]_{-z}!}$$

where $[n]_z = (e^{z n} - 1)/(e^z - 1)$ and $[n]_z! = [n]_z [n-1]_z \cdots [2]_z [1]_z$.

The universal T matrix is then [1]

$$T = e_z^{\phi \otimes F} \, e^{\gamma \otimes J_3} \, e_{-z}^{\eta \otimes E} \tag{13}$$

where $e_z^A = \sum_i \frac{A^i}{[i]_z!}$ (to compare with [1] $q = q' = e^{z/2}$).

The content of (5) can now be made explicit. Indeed, if for any element $\xi \in Fun_q(SU(2))$ we resume the initial notations $\xi' = \xi \otimes 1$, $\xi'' = 1 \otimes \xi$ and we rewrite (11) accordingly, we get the suggestive result

$$\left(e_z^{\phi' \otimes F}\, e^{\gamma' \otimes J_3}\, e_{-z}^{\eta' \otimes E}\right)\left(e_z^{\phi'' \otimes F}\, e^{\gamma'' \otimes J_3}\, e_{-z}^{\eta'' \otimes E}\right) = e_z^{m^*(\phi) \otimes F}\, e^{m^*(\gamma) \otimes J_3}\, e_{-z}^{m^*(\eta) \otimes E}$$

$$= e_z^{(\phi' + e^{-\gamma'/2}\phi''(1+\eta'\phi'')^{-1} e^{-\gamma'/2}) \otimes F}\, e^{(\gamma' + \gamma'' - 2z \sum_n (-\eta'\phi'')^n / (1-e^{-zn})) \otimes J_3}$$

$$\times e_{-z}^{(\eta'' + e^{-\gamma''/2}(1+\eta'\phi'')^{-1}\eta' e^{-\gamma''/2}) \otimes E} .\tag{14}$$

Obviously, in the limit $z \to 0$ we recover the exponential mapping and the Lie group multiplication. Moreover (14) gives a very neat example of the conditions posed by the Friedrichs theorem [8]: indeed the non-commutativity of $Fun_q(SU(2))$ and the presence of non-standard exponentials is needed to compensate the fact that the generators of the quantum algebra are no longer primitive. Additional peculiarities are also connected with the antipode, but we shall present them for the Heisenberg quantum group with the explicit calculations, which, in that case, are much simpler. We finally observe that the expression for $m^*(\gamma)$ depends on z and that the limit $z \to 0$ gives the classical composition law

$$\lim_{z \to 0} m^*(\gamma) = 1 \otimes \gamma + \gamma \otimes 1 + 2\log(1 \otimes 1 + \eta \otimes \phi).$$

3. The exponential mapping for $H_q(1)$

In [5] a quantum deformation of the Heisenberg group has been determined. With a slight change in the definitions with respect to [5] the commutation relations of the generators a_-, a_+ and H of the quantum Heisenberg algebra can be written in the form

$$[a_-, a_+] = \frac{\sinh(w H)}{w} \qquad [H, \cdot] = 0.$$

The corresponding coproducts read

$$\Delta(a_-) = 1 \otimes a_- + a_- \otimes e^{w H} \qquad \Delta(a_+) = e^{-w H} \otimes a_+ + a_+ \otimes 1$$

H being primitive, while the antipodes are

$$S(a_-) = -e^{w H} a_- \qquad S(a_+) = -e^{-w H} a_+ \qquad S(H) = -H.$$

In order to determine the dual structure $Fun_q(H(1))$ we consider the generators α, β, δ satisfying $\langle \alpha, a_- \rangle = 1$ $\langle \beta, H \rangle = 1$ $\langle \delta, a_+ \rangle = 1$ with commutation relations, coproducts and antipodes given by:

$$[\alpha, \delta] = 0 \qquad [\beta, \alpha] = -w\,\alpha \qquad [\beta, \delta] = -w\,\delta$$

$$m^*(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha \qquad m^*(\beta) = \beta \otimes 1 + 1 \otimes \beta + \alpha \otimes \delta$$

$$m^*(\delta) = \delta \otimes 1 + 1 \otimes \delta \tag{15}$$

$$S^*(\alpha) = -\alpha \qquad S^*(\beta) = -\beta + \alpha\delta \qquad S^*(\delta) = -\delta.$$

A direct calculation shows that $\langle \delta^a \, \beta^b \, \alpha^c, a_+^d \, H^e \, a_-^f \rangle = a! \, \delta_{ad} \, b! \, \delta_{be} \, c! \, \delta_{cf}$, so that the T matrix results in

$$T = e^{\delta \otimes a_+} \, e^{\beta \otimes H} \, e^{\alpha \otimes a_-} \, .$$

Unlike the case of $SU_q(2)$ and similarly to what occurs for Lie algebras, the universal T matrix is now expressed in terms of simple exponentials: however, as already observed at the end of the previous section, the non-primitive nature of the quantum generators a_-, a_+ and H must be compensated by the non-vanishing commutators of the C^*-algebra elements δ, α and β in order to reproduce the same 'group composition' of the coordinates as given in (15). Let us also show directly that the inverse of the universal T matrix cannot be expressed in terms of exponentials of the antipodes: indeed, from (6),

$$T^{-1} = S^* \left(\frac{\delta^a}{a!} \, \frac{\beta^b}{b!} \, \frac{\alpha^c}{c!} \right) \otimes a_+^a \, H^b \, a_-^c = \frac{1}{c!} S^*(\alpha)^c \, \frac{1}{b!} S^*(\beta)^b \, \frac{1}{a!} S^*(\delta)^a \otimes a_+^a \, H^b \, a_-^c$$

$$= \frac{\delta^a}{a!} \, \frac{\beta^b}{b!} \, \frac{\alpha^c}{c!} \otimes S(a_+^a \, H^b \, a_-^c) = \frac{\delta^a}{a!} \, \frac{\beta^b}{b!} \, \frac{\alpha^c}{c!} \otimes S(a_-)^c \, S(H)^b \, S(a_+)^a \, .$$

If we re-order the terms in this expressions to reconstruct an exponential series, we find

$$T^{-1} = e^{-\alpha \otimes a_-} \, e^{-\beta \otimes H} \, e^{-\delta \otimes a_+}$$

namely the obvious expression that, however, does not have the chosen ordering of the three exponential factors: this, indeed, is related to the deep question of what should be taken as a quantum analogue of the classical Baker–Campbell–Hausdorff formula [10].

It was shown in [5] that the results for $H_q(1)$ can be obtained by contracting the quantum algebra $SU_q(2)$. It is interesting to observe that this procedure also holds at the level of the quantum group $Fun_q(SU(2))$ and therefore the T matrix itself can be obtained by contraction. Indeed, the rescaling $a_- = \varepsilon^{1/2} \, E$, $a_+ = \varepsilon^{1/2} \, F$, $H = \varepsilon \, 2J_3$, with $w = \varepsilon^{-1} z/2$ reproduces, in the limit $\varepsilon \to 0$ the quantum algebra $H_q(1)$. In order to maintain the pairing relations (9) we have to define $\alpha = \varepsilon^{-1/2} \eta$, $\beta = \varepsilon^{-1} \gamma/2$ and $\delta = \varepsilon^{-1/2} \phi$: the relations of $Fun_q(H(1))$ are simply obtained by using this rescaling on $Fun_q(SU(2))$ and taking $\varepsilon \to 0$. The T matrix for $H_q(1)$ is then calculated from that of $SU_q(2)$ by taking the limit $\varepsilon \to 0$: because of this limit it is clear that the q-exponentials become usual exponentials.

4. The case of Euclidean and Galilei quantum groups

In this last section we shall determine the T matrices for two inhomogeneous quantum groups, the Euclidean quantum group $E_q(2)$ and the Galilei $\Gamma_q(1)$.

The quantizations of $E(2)$ have been thoroughly discussed in [9]. Here we shall be concerned with that quantum deformation, initially found in [6], which can very simply be obtained by a contraction of the $SU_q(2)$ algebra, rescaling the generators as

$$P_+ = \varepsilon \, J_+ \qquad P_- = \varepsilon \, J_- \qquad J = J_3$$

and taking the limit $\varepsilon \to 0$.

If we define the new basis (b_-, J, b_+), with

$$b_- = e^{-zJ/2} \, P_- = \varepsilon F \qquad b_+ = e^{zJ/2} \, P_+ = \varepsilon E$$

and denote by (π_-, π, π_+) the dual basis of (b_-, J, b_+), the relations

$$\langle \pi_-, b_- \rangle = 1 = \langle \pi_-, \varepsilon\, F \rangle = \langle \varepsilon \pi_-, F \rangle \qquad \langle \pi, J \rangle = 1 = \langle \pi, J_3 \rangle$$

$$\langle \pi_+, b_+ \rangle = 1 = \langle \pi_+, \varepsilon\, E \rangle = \langle \varepsilon \pi_+, E \rangle \,.$$

imply that $\pi_- = \varepsilon^{-1} \phi$, $\pi = \gamma$, $\pi_+ = \varepsilon^{-1} \eta$, where (ϕ, γ, η) have been introduced in section 2. By means of these rescalings we can directly contract the relations of $Fun_q\,(SU(2))$ obtaining

$$[\pi_-, \pi_+] = 0 \qquad [\pi, \pi_-] = -z\,\pi_- \qquad [\pi, \pi_+] = -z\,\pi_+ \,. \tag{16}$$

Let us notice that the algebra relations in (10), (15) and (16) all have the same structure.

In the limit $\varepsilon \to 0$ the coproducts read

$$m^*(\pi_-) = \pi_- \otimes 1 + e^{-\pi} \otimes \pi_- \qquad m^*(\pi) = 1 \otimes \pi + \pi \otimes 1 \qquad m^*(\pi_+) = 1 \otimes \pi_+ + \pi_+ \otimes e^{-\pi}$$

and the antipodes

$$S^*(\pi_-) = -e^{\pi}\, \pi_- \qquad S^*(\pi) = -\pi \qquad S^*(\pi_+) = -\pi_+ e^{\pi} \,.$$

It is not difficult to show that this structure is equivalent to the one found in [4, 9, 11]. It allows, however, a very plain determination of the universal T matrix. Indeed, contracting the expression (13) we find

$$T = e_z^{\pi_- \otimes b_-}\, e^{\pi \otimes J}\, e_{-z}^{\pi_+ \otimes b_+} \,. \tag{17}$$

Using the fact that

$$e_z^A\, e_z^B = e_z^{A+B} \qquad \text{if} \quad AB = e^{-z} BA$$

all the properties of T can be directly verified.

Let us finally analyse the results for the deformation of the one-dimensional Galilei group, $\Gamma_q(1)$, which, as shown in [7], has remarkable physical applications since it describes the dynamical symmetries of magnon systems on a linear lattice. Contrary to what occurs for $E_q(2)$, the quantum group $\Gamma_q(1)$ cannot be obtained from a contraction: therefore, in order to find the expression for the T matrix we have to determine the explicit duality relations.

The Hopf algebra of $\Gamma_q(1)$, [7], is defined by the commutation relations

$$[B, P] = iM \qquad [B, T] = i/a\, \sin(aP) \qquad [P, T] = 0 \qquad [M, \cdot] = 0$$

with coproducts, antipodes and co-units

$$\Delta B = e^{-iaP} \otimes B + B \otimes e^{iaP} \qquad \Delta M = e^{-iaP} \otimes M + B \otimes e^{iaP}$$

$$\Delta P = 1 \otimes P + P \otimes 1 \qquad \Delta T = 1 \otimes T + T \otimes 1$$

$$S(T) = -T \qquad S(B) = -B - aM \qquad S(P) = -P \qquad S(M) = -M \,.$$

If we define the pairing

$$\langle \mu, m \rangle = \langle x, P \rangle = \langle t, T \rangle = \langle v, b \rangle = 1$$

where $m = e^{-iaP}M$ and $b = e^{iaP}B$, we get the relations for $Fun_q\,(\Gamma(1))$

$$[v, x] = 2iav \qquad [v, \mu] = -av^2 \qquad [x, \mu] = -2ia\mu \qquad [t, .] = 0$$

while the coproducts, antipodes and co-units are as in the Lie case, namely

$$\Delta v = 1 \otimes v + v \otimes 1 \qquad \Delta\mu = 1 \otimes \mu + \mu \otimes 1 + i\,v \otimes x - 1/2\,v^2 \otimes t$$

$$\Delta t = 1 \otimes t + t \otimes 1 \qquad \Delta x = 1 \otimes x + x \otimes 1 + i\,v \otimes t\,.$$

Defining the basis of $\Gamma_q(1)$ as $X_{abcd} = m^a P^b T^c b^d$ the dual basis of $Fun_q\,(\Gamma(1))$ is $(\mu^a x^b t^c v^d)/(a\,!\,b\,!\,c\,!\,d\,!)$, so that, as for the Heisenberg group, the universal T matrix is given in terms of simple exponential series

$$T = e^{\mu \otimes m}\,e^{x \otimes P}\,e^{t \otimes T}\,e^{v \otimes b}\,.$$

To conclude, we can mention that universal T matrices for other inhomogeneous quantum groups, such as the singular deformation of $E(2)$, [9], or the three-dimensional Euclidean group, [5], can be obtained along the same lines. Obviously, in the presence of an R-matrix, the duality relations are more easily determined. This is, for instance, the case of $Fun_q\,(E(3))$, whose relations†, in the notations of [5],

$$[z, \tilde{x}] = -w\tilde{x} \qquad [\theta, \tilde{x}] = w\,\sin\theta\,\tan(\theta/2) \qquad [\tilde{y}, \tilde{x}] = w\tilde{y}\,\tan(\theta/2)$$

$$[\theta, z] = w\,\sin\theta \qquad [\tilde{y}, z] = w\tilde{y} \qquad [\omega, \tilde{y}] = -2w\,\tan(\theta/2)$$

are defined in terms of (7).

5. Concluding remarks

Starting from the definition of the universal T matrix given in [4], we have stressed its direct connection with the exponential mapping for Lie groups. The T matrix collects both the structure of the quantum algebra and of its dual and allows an algebraic formulation of the standard 'RTT' relations. Its expression in terms of q-exponentials, as already given in [1], is found to reduce to the usual exponentials for those inhomogeneous groups for which the quantum parameter can be re-absorbed by a new definition of the generators of the quantum algebra [5, 9]: when this occurs, as for $E_\ell(2)$ and $\Gamma_q(1)$, the parameter acquires a physical dimension and is naturally interpreted as a lattice spacing. Therefore, the universal T matrix, regarded in [4] as the quantum transfer matrix in models of $(1 + 1)$ lattice field theory, can also naturally be used for the study of the quantum deformation of group properties of physical systems with kinematical symmetries generated by quantum algebras, as in the cases of magnons and phonons [7, 12]. In this context the explicit structure for all the semisimple quantum groups and for the other ones relevant for applications (e.g. q-Poincaré) deserves a careful examination. Moreover, the use of those results in the study of non-commutative geometry and q-special function should also be relevant.

† Here we take the opportunity of correcting a misprint of [5].

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